ON THE NON-PERTURBATIVE PROPERTIES OF THE YANG-MILLS

VACUUM AND THE VACUUM ENERGY DENSITY

V.G. KSENZOV

Institute of Theoretical and Experimental Physics,
Moscow,Russia

A b s t r a c t

The non-perturbative part of the vacuum energy density for static configurations in pure $SU(2)$ Y-M theory is described. The vacuum state is constructed.

e-mail:ksenzov@vxitep.itep.ru

1 Introduction

The goal of the present paper is to investigate non-perturbative dynamics in the pure Y-M theory which is the key to the calculation of the vacuum condensates.

To understand the actual dynamics and the role of the non-perturbative effects we should have an explicit form of the non-perturbative fields. As a rule the non-perturbative effects are associated with classical fields characterized by topological charges. To see how the non- perturbative fluctuations generate physical amplitudes we shall treat the non-perturbative part of the vacuum expectation value (v.e.v.) of the energy-momentum tensor $\theta_{\mu\mu}$ in Y-M theory in four dimensions. In the case of theories without dimensional parameters the v.e.v. of $\theta_{\mu\mu}$ gives the vacuum energy density ε_{vac} and the characteristic mass scale thus defining properties of the effective theory [1]. It has been noticed that the connection of non-perturbative effects with classical fields is not always straightforward. Indeed, there is a set of models, such as $O(N)$ non-linear sigma models ($N > 3$) which do not have topological solutions, but do have non-perturbative effects which have been obtained within perturbation theory [1, 2, 3]. Besides, the physical vacuum has no topological charge ($Q = 0$) by definition but we believe its structure has a non-perturbative nature.

There are several methods of calculation of the non-perturbative contributions into v.e.v.'s of different correlations. A solution of the problem in Y-M theory have been suggested by 't Hooft [4] and was a subject of intense study. However, there are several problems arising within this approach when instanton background is considered in the dilute-gas approximation [5]. One of them is that contributions to the integrals over the instanton size come from the region of large sizes where the initial approximation is no longer valid. Therefore, the exact magnitude of the non-perturbative effects remains unknown. Recently an alternative method has been proposed in which the non-perturbative dynamics has been investigated for the example of the two dimensional $O(N)$ non-linear sigma model in the large N limit [6, 7]. In this model any instanton effects are absent. Here we shall use the basic ideas which have been developed in these papers. For this reason we recall the main features of the approach.

It was shown that fluctuations describing the vacuum properties of a theory are subject to the requirements for potential energy to be in the minimum and for conjugated canonical momenta to be zero. Such fluctu-

ations are not operators but a c-number function. For similar reason the temporal component of the gauge field are c-number functions.

It is evident that constant fields may obey these conditions. In such a case the quantum fluctuations around the constant background describe the perturbative properties of the theory. If non-constant fields satisfy these conditions then the quantum fluctuations around the non-constant background describe the non-perturbative properties of the theory.

The regularization procedure is essential for calculation of vacuum condensates. The regularization by separating the quantum fields in different points is used.To this end the point of the space in which the regularization procedure has to be done should be replaced by a sphere S^2 having a small radius r which is set to zero at the very end of the calculation. The quantum field is defined on the surface of the sphere. From dimensional consideration ε_{vac} has to be proportional to $1/r^4$, therefore, naive ε_{vac} goes to infinity when r tends to zero. In real fact the non-perturbative value of ε_{vac} is finite owing to quantum effects.

The non-trivial classical fields, mentioned above, are characterized by the topological charge Q. The energy funtional for static non-trivial configuration is defined as $E = 4\pi Q/g^2$, where g is the coupling constant. Obviously that the physical vacuum has no topological charge and may be presented as a sum of classical configurations which contribute to the topological charge with different signs. However, there is an alternative possibility. If the physical space has two boundaries, the topological charge may be equal to zero for non-trivial classical configurations. Such situation emerges due to the suggested regularization method. The point is that introduction of the sphere of small radius gives one more boundary (another boundary is at large distance) and the topological charge is given by the difference of contributions from the boundaries and is equal to zero for the infinitely small radius [8] say, for the radius which is inverse to the mass of the ultraviolet cutoff.

In the present paper we shall adop the basic ideas which are obtained in sigma models to investigate the non-perturbative dynamics and to calculate the vacuum energy density in Y-M theory. In sect.2 the non-perturbative structure of vacuum state is discussed in general form. We describe the classical fields and discuss their properties in sect.3. In sect.4 we calculate the vacuum energy density. It is necessary to stress that the vacuum state is not obtained by solving the Schodinger equation. We belive that the constructed vacuum state is one of a coherent type.

2 The non-perturbative structure of the vacuum state.

In the present section we suggest the method of construting of the non-perturbative vacuum state using perturbative vacuun state. Our basic idea can be better explained in terms of the quantum mechanics.

Let us assume that $\Psi(x)$ is the ground state of some quantum system and T is the translation operator. We know that the average value of the quantity $x + x_0 = T^{-1}xT$ is determined by the following expresion

$$\bar{x} = \int_{-\infty}^{\infty} dx \Psi^*(x) \left(x + x_0\right) \Psi(x) = \int_{-\infty}^{\infty} dx \, \left(T\Psi\right)^* \, x \, \left(T\Psi\right) = x_0.$$

This example shows we do not need to know, generally speaking, the explicit ground-state $\Psi(x - x_0)$ to obtain the average value of $\bar{x} \neq 0$. We can get it knowing only the translation operator and the parameter of the transformation x_0. Notice that $\Psi(x)$ and $\Psi(x - x_0)$ do not satisfy the requirement of orthogonality. If the anologous situation takes place in quantum field theory then having the non-perturbative fluctuation and some translation operator one can construct the vacuum state and to calculate vacuum condensate. However, contrary to quantum mechanics one has to know some starting ground state in field theory because of v.e.v. is obtained by another mean.It turns out that perturbative ground state may be selected as the starting state. This situation facilitates the problem.

It was shown in the framework of the sigma models in two dimensions that this idea leads to the correct result for the vacuum condensate of $\theta_{\mu\mu}$ [7]. Therefore we are about to discuss how the idea is realized in quantum field theory.

Let us show how the non-perturbative state is constructed. At the beginning the non-perturbative vacuum fields fluctuations are defined and then the translation operator and the non-perturbative vacuum state is constructed.

Assume that L is the Lagrangian of some field system and h is the Hamiltonian density. There is a relation

$$L = \dot{\phi}\pi_0 - h,$$

where ϕ is the field and $\pi_\mu = \delta L/\delta\partial_\mu\phi$ is the canonical momentum. Here group indexes are omitted. The Hamiltonian is defined as

$$H = \int dx^{n-1} \, (\pi_\mu^2 + V(\phi)).$$

Let us assume that the potential gets the minimum $V(\phi_v) = 0$ when $\phi_v^2(x) = const$. The finite energy condition is satisfied if $\pi_\mu = 0, V(\phi) = 0$ everywhere at large distances. The solutions satisfying this condition are a set of the vacuum fields because we believe that fields take their vacuum value at large distances in theory without sources. It is believed that vacuum is costructed in the same way at any point of the physical space. Therefore the condition defines the vacuum field in any point of the physical space except a singular point. The vacuum field can not be quantizated due to the fact that their conjugated canonical momenta are equal to zero $\pi_0 = 0$ and $\phi_v(x)$ is c-number function.

If the canonical momentum is $\pi_\mu = \partial_\mu \phi$, then the solutions of the condition are trivial $\phi_v = const$. In this case the theory is quantized around constant background and the quantum fluctuations describe the perturbative properties of the theory [7].

If the canonical momentum involves a gauge field A_μ,i.e. $\pi_\mu = \partial_\mu \phi + g A_\mu \phi$,then there are solutions of the condition $\pi_\mu = o$ which are topologically trivial $A_\mu = 0, \phi_0 = const$ and topologically non-trivial $A_\mu = 1/g U \partial_\mu U^{-1}, \phi_v = U \phi_0$ at large distances. U(x) is an arbitrary element of a gauge group and must be non-singular only on a bondary. The fields $\phi_v(x)$ are c-number functions only if we quantize the theory over the non-constant background $\pi_0 = \partial_o \phi_v + U \partial_\mu U^{-1} \phi_v = 0$. In the perturbation theory, when $A_\mu = 0$ on the boundary and non-singular in all physical space, the canonical momentum $\pi_\mu = \partial_\mu \phi \neq 0$ therefore the fields $\phi_v(x)$ are quantum fields which describe the non-perturbative properties of the theory as will be argued in the paper. As discussed in [7] this is the reason why some results which are obtained in the perturbative methods are related to the non-perturbative effects.

Let us suppose we quantize our theory over the constant background. If the constant is equal to zero then the vacuum of the theory may be described in a usual way as the eigenfunction of the annihilation operator a_0 with zero eigenvalue of the momentum $a_0|0>= 0$ [12] and v.e.v. of $\phi(x)$ is $< 0|\phi(x)|0 >= 0$. If one wants to have $< 0|\phi(x)|0 >\neq 0$ then a new vacuum state have to be constructed. The operator which carries out the transformation $\phi \rightarrow v$ is well known to be the operator of the canonical

momentum. The translation operator is

$$T = \exp v \int dx\, \partial_0 \phi = \exp v(a_0 - a_0^+)$$

and the new vacuum state is written as $|v >= T|0 >$. In this case the annihilation operator a_0 has no zero eigenvalue $a_0|v >= v|v >$ and v.e.v. of $\phi(x)$ is $< v|\phi(x)|v >= v$ [12].

If the vacuum fluctuation is a function then the translation operator can be written as

$$T = \exp \int dx^{n-1} \left(\phi_v(x)\pi_0 \,+\, \partial_0\phi_v(x)\phi(x)\right).$$

Here the last term in the integral generates the translation of the canonical momentum into $\partial_0\phi_v(x)$.

Considering v.e.v. of some operator in the non-perturbative vacuum state $|\phi_v >= T|0 >$ we obtain that all fields operators are substituted on the vacuum fluctuation and then v.e.v. have to be calculated using the same method of regularization as is given below. Notice that the vacuum state $T|0 >$ is the total vacuum state which contains perturbative part $|0 >$ and non-perturbative part $(T-1)|0 >$ of the vacuum state therefore there is no orthogonality between the perturbative and the total vacuum states.

Here we recall that if there is a symmetry in the theory then the symmetry generator Q annihilates the vacuum state $Q|0 >= 0$. The condition of the symmetry breaking is the absence of the symmetry invariance of the vacuum state $Q|0 >\neq 0$. In is necessary to stress that the symmetry generator is common one but the vacuum states are different states. If we should like to obtain the vacuum energy changes then Hamiltonian have to be precisely the same in the vacuum states.

The anologous situation takes place in this case. By choosing as the starting state the vacuum state of the perturbative theory we have to use free Hamiltonian of the theory and non-perturbative vacuum state to obtain vacuum energy changes.

However there is a problem arising within this approach in Y-M theory. The point is that free Hamiltonian of the gauge fields is not invariant under SU(2) local gauge transformation corresponding to Y-M theory. Therefore the vacuum energy which is obtained by this the method is gauge dependent.

It is known [1] that the vacuum energy may be decomposed into non-perturbative and perturbative parts. The pertubative part is scale depen-

dent. The non-perturbative part is scale independent. Besids the non-perturbative part of the vacuum energy is defined on the non-trivial topological fields because the physical local gauge transformations which are trivial from the point of view of topology can not break down the non-perturbative results. The gauge transformations change only the perturbative part of the vacuum energy which should be subtracted because it has no physical meaning.

Notice that the vacuum energy density is usually obtained as v.e.v. of the trace of the energy-momentum tensor which have to be defined in gauge non-invariant way according to the above arguments.

The translation operator in Y-M theory is definefed as

$$T \exp \int dx^3 J_i^a \, E_i^a$$

in the case of the static vacuum cofigurations. Using this vacuum state we can obtain v.e.v. of $\theta_{\mu\mu}$ by the operator method.However for pedagogical purpose the calculation of the vacuum energy density in Y-M theory is carried out using functional integral.

3 Classical vacuum in Yang-Mills theory

The structure of the vacuum in the pure Y-M theory was well studied. An extensive reference list can be find in [8]. Here we are interested in static properties of the vacuum. At the begining we will remind the formulation given in paper [9]. To set out notations we briefly recapitulate some results relevant to our work.

The Hamiltonian of the classical Y-M theory in Euclidian space is

$$H = \frac{1}{2} \int d^3x \, (E_k^a)^2 + (B_k^a)^2 \,, \tag{1}$$

where $E_k^a = F_{k4}^a$, $B_k^a = 1/2 \varepsilon_{klm} F_{lm}^a$, $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g\varepsilon^{abc} A_i^a A_j^c$; a, b, c are the group indices; $k, l, m = 1, 2, 3$. Let the gauge group be $SU\,(2)$ and $(t^a)^{bc} = \varepsilon^{abc}$ be generators of the group $SU\,(2)$ in the adjoint representation. In this case the fields A_μ^a are real. We also imply that the gauge fields are subject to the first class constraint and there is the gauge fixing condition $\partial_\mu A_\mu = 0$.

The gauge fields which describe the classical properties of the vacuum must satisfy the condition $H = 0$ in the static case. This, in its turn, yields

$$F_{4k}^a = F_{lm}^a = 0, \tag{2}$$

i.e. the canonical momenta are equal to zero. It follows that the fields which describe the classical vacuum of the theory are pure gauge.

The space components of the potential A are

$$A_i = \frac{1}{g} U(x) \partial_i U^T(x), \tag{3}$$

where $U(x)$ is an arbitrary element of the gauge group which is taken independent of x_4 and $U(x)U(x)^T = 1$. Here it is necessary to stress that the gauge field (3) may be eliminated by a gauge transformation only if $U(x)$ goes to zero at large distances. However we shall also deal with non-trivial topological configurations therefore the function $U(x)$ must be non-singular on the boundary and corresponds to topological charge. As has already been mentioned, it is possible that the total topological charge and energy functional are equal to zero even when there are non-trivial topological fields. Such possibility is discussed in the next section. In this case A_i (3) can not be eliminated by gauge transformations [8].

The components of the gauge field A_4 are usually chosen equal to zero but, as shown in [9], $A_4 \neq 0$ defines the topological charge of the monopole and, as we shall see later, other non-perturbative properties of the theory in the static case.

The static configurations A_4^a can be obtained from the condition (2)

$$F_{4k}^a = \partial_k A_4^a + g\varepsilon^{abc} A_k^b A_4^c = 0. \tag{4}$$

Multiplying this equation by A_4^a and taking into account antisymmetricity of the tensor ε^{abc}, we get

$$\partial_k \left(A_4^a\right)^2 = 0, i.e. \left(A_4^a\right)^2 = const. \tag{5}$$

The constant in (5) is arbitrary but it is important that it may be chosen non-zero [9]. Although the classical theory has no dimensional parameter we can introduce arbitrary quantity μ due to the condition (5) and define

$$A_4^a(x) = \mu \Phi^a(x), \tag{6}$$

where Φ^a is a dimensionless function which is subject to the condition

$$\Phi^a(x)\Phi^a(x) = 1. \tag{7}$$

It follows from (7) that the fields $\Phi^a(x)$ can be represented as

$$\Phi(x) = U(x)\Phi_0, \tag{8}$$

where Φ_0 is a constant vector in the colour space. As follows from eq.(8) the shift of ϕ in x-space is equivalent to the rotation of ϕ_0 in the colour space.

The vector potential A_k^a may be expressed by a combination of the fields $\Phi^a(x)$. Multiplying eq.(4) by $\varepsilon^{ial}\Phi^l$ we obtain

$$J_k^i = g\left(\delta^{in} - \Phi^i\Phi^n\right) A_k^n, \tag{9}$$

where $J_k^i = \varepsilon^{ial}\partial_k\Phi^a\Phi^l$. Introducing $P^{in} = \delta^{in} - \Phi^i\Phi^n$ then $P^{in}\Phi^n = 0$ and $P^{in}P^{nl} = P^{il}$. Therefore multiplying (9) by P^{if} we get

$$P^{if}\left(J_k^f - gA_k^f\right) = 0. \tag{10}$$

The solution of eq. (10) is

$$A_k^a = \frac{1}{g}\left(J_k^a - n_k\Phi^a\right), \tag{11}$$

where $n_k(x)$ is an arbitrary function. The solution (11) also satisfies the condition $F_{ij} = 0$ (2) if $n_k(x)$ is a pure gauge in the group $U(1)$, i.e. $\partial_i n_j - \partial_j n_i = 0$. The quantity n_i defines the projection of A_k^a onto $\Phi^a(x)$. Indeed, from (11) we get

$$n_k = -gA_k^a\Phi^a(x) \tag{12}$$

J_i^a is a conserved current ($\partial_i J_i^a = 0$) associated with the global symmetry $SO(3)$ of the theory. The conservation of the current follows from the relation

$$\partial^2\Phi^a = \Phi^a\left(\partial_i\Phi^a\right)^2,$$

which can be obtained from eq.(7).

The gauge fixing condition $\partial_\mu A_\mu = \partial_k A_k = 0$ may be satisfied by requiring the orthogonality of Φ^a and A_k^a or $n_k = 0$, i.e. n_k is a gauge-fixing parameter. In such a case the gauge potential (11) is not fixed by the Coulomb gauge. The property of the Coulomb gauge was first discussed by V.N. Gribov [10].

Let us find how the fields $\Phi^a(x)$ are expressed in terms of the current J_k^a. To do this we may use eq. (4). Substituting (6) and(11) into (4) we obtain

$$\partial_k\Phi^a(x) + \varepsilon^{abc}J_k^b(x)\Phi^c(x) = 0. \tag{13}$$

The solution of eq.(13) can be written as

$$\Phi(x) = P\exp\left(-\int\limits_{x_0}^{x} dz_i\, J_i\right)\Phi(x_0)\ . \tag{14}$$

where $J_i = J_i^a t^a$ and x_0 is an arbitrary point. As it is clear from (14) and (7) the values of the field $\Phi(x)$ in two different fixed points x_1 and x_2 are related to each other by the global group transformation and any invariant of the global transformation is independent of choosing the point x_0. One can see from (3) and(11) that the current J_k is definded as

$$J_k = U\partial_k U^{-1}, \tag{15}$$

when $n_k = 0$. Therefore the integral (14) does not depend on the integration path if the function $U(x)$ has no singularities. However to have non-trivial topological effects we should also admit the occurrence of gauge functions which have a singularity. Then the function (14) is a pure gauge at any region which does not contain the singular points. However it is ambiguous in the whole space therefore globally the fields J_k are not a pure gauge. We recall that the integral $\oint dz_i\, \hat{J}_i$ taken along a closed contour encircling the singularity is not equal to zero, i.e. depends on the integration contour.

4 On the calculation of the vacuum energy density.

The vacuum energy density is defined as

$$\varepsilon_{vac} = \frac{1}{4} < 0|\theta_{\mu\mu}|0 >, \tag{16}$$

where $\theta_{\mu\mu}$ is the trace of the energy-momentum tensor, the numerical coefficient is determined by the dimension of the physical space $D = 4$.The vacuum energy density is defined as a sum of the perturbative and non-perturbative contributions. Since we are interested in the non-perturbative effects we will now assume that the perturbative contributions into the vacuum energy density are subtracted from ε_{vac} in eq. (16). The non-perturbative part of the vacuum energe density is denoted by ε^n_{vac}.

It is well known that the dilatation anomaly in gauge theories is proportional to the β-functon and the field strength squared but the scale independent value of the dilatation anomaly is only $g^2(F^a_{\mu\nu})^2$. Therefore only this part ε^n_{vac} has a physical significance. For this reason we shall calculate the vacuum condensate of $g^2(F^a_{\mu\nu})^2$. Notice that this method seems to avoid the computation of the dilatation anomaly condensate if renormalization constants are known at all orders.

Let the Lagrangian be defined in terms of the renormalisation gauge fields and constants as

$$L=-\frac{1}{4}\left(F^a_{\mu\nu}\right)^2-\frac{1}{2\alpha}\left(\partial_\mu A^a_\mu\right)^2-\frac{1}{4}\left(Z_3-1\right)\left(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu\right)^2. \qquad (17)$$

The ghost are omitted because they are essential only in perturbative calculations. The gauge fields, the gauge fixing parameter and the coupling constant are given by

$$A_b=Z_3^{1/2}A, \alpha_b=Z_3\alpha, g_b=Z_g g,$$

where A_b ,g_b ,α_b are bare quantities. The quantities Z_g and Z_3 are calculated at the one-loop level. In such a case we have

$$Z_3=1+\frac{g_b^2}{16\pi^2}\left(\frac{13}{6}N-\frac{\alpha_b}{2}\right)\ln M^2/\mu^2, Z_g=1+\frac{g_b^2}{16\pi^2}\frac{13}{6}N\ln M^2/\mu^2, \quad (18)$$

where μ^2 is a normalization point, M^2 is an ultraviolet cutoff.

The last term in eq.(17) is a counterterm which is not gauge invariant. As above gauge invariance of the counterterm is not required.

It is known [11] that the conformal anomaly $\theta_{\mu\mu}$ can be obtained as the variation of action when the ultraviolet cutoff M^2 changes into $(1+\eta)M^2$ with the coupling constant kept fixed. Here η is the parameter of the global scaling transformation. Then we can obtain

$$\theta_{\mu\mu}=-\frac{g_b^2}{64\pi^2}b\left(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu\right)^2. \qquad (19)$$

Here $b=13/6N-\alpha_b/2$.

In terms of the generating functional Z_E in Euclidian space, v.e.v. of $\theta\mu\mu$ can be written as

$$<0|\theta_{\mu\mu}|0>=Z_E^{-1}\int DA_i\,\theta^R_{\mu\mu}\,exp^{-S_E} \qquad (20)$$

Here the quantity $\theta^R_{\mu\mu}$ have been obtained from eq.(19) by introducing the regularization.

Let us calculate the vacuum energy density at some point x_0. Let us assume that x_0 is enclosed by a sphere S^2 of a small radius r. To regularize the quantity $\theta_{\mu\mu}$ we define the quantum fields in different points on the surface of the sphere S^2. Let us put the radius to be inverse of the parameter μ from eq.(6).

Notice that making so we spoil the gauge symmetry. It should be restored if the operator is the gauge invariant quantity. However,since the

operator $\theta_{\mu\mu}$ eq.(19) is not gauge invariant and from the very beginning, we may not take care of the gauge symmetry at all. Also it should be kept in mind that the quantity A_4^a are c-number fields and therefore the quantity $(\partial_k A_4^a)^2$ do not have to be regularized. As will be shown later therefore the contributions of the electrical strengh in the non-pertrubative vacuum energy are absent for the static fields.

According to V.N. Gribov [10], the contributions of the configurations J_i^a can not be compensated by the ghost fields because the Faddeev-Popov determinat has zero and the standart method has to be improved. In the static case the action has a minimum on this configurations as pure gauge configurations in all physical space besides the point of the singularity. Here we keep in mind that contributions to the integral which come from the sphere rigion are ignored,but the contributions from the surface of the sphere are.

At the begining one may speculate that the singular point is x_0 where the operator $\theta_{\mu\mu}$ is defined. We recall that the point x_0 is surrounded by the sphere of a small radius and area bounded by the sphere surface is excluded from consideration. In such case the topological charge equals to zero. Really, the topological charge is

$$Q = \oint_{S_\infty} d\sigma_k \, j_k - \oint_{s_r} d\sigma_k \, j_k = 0, \tag{21}$$

where $j_k = \varepsilon_{kij} \partial_i A_j^a \Phi^a$, S_∞ and s_r are the respective surfaces of the sphere at large and small distances. The result (21) is valid for any infinitely small quantity r, which has to be considered as a "physical" zero. The result (21) is explained by the absence of the singularity in the region between two sphere surfaces where the fluctuations are defined.

At first glance it would seem that we discussed a special case because the singular point coincides with the point in which the operator $\theta_{\mu\mu}$ is examined. Indeed, this is not the case. As the function $\Phi^a(x)$ are subject to constraint (7) which is independent of a position of the singular point and the quantity $(\partial_i \Phi^a)^2$ is a translational invariant function therefore we can give any position to the singularity for any combination being quadratic in Φ^a or $\partial_i \Phi^a$. As we shall see, the v.e.v. of $\theta_{\mu\mu}$ may be written as one of such combinations due to $\theta_{\mu\mu}$ is independent of the position of the singularity.

We recall that the action has the minimum on the fields $A_i^a = \frac{1}{g} J_i^a$

therefore v.e.v. of $\theta_{\mu\mu}$ can be written as follows

$$
\begin{aligned}
< 0|\theta_{\mu\mu}|0 >= \lim_{\Delta\to 0} -g_b^2 b/64\pi^2 \left[2\mu^2 \left(\partial_i \Phi^a\right)^2 - 1/g_b^2 \left(\partial_i J_j^a(x_0-\Delta) - \right.\right. \\
\left.\left.\partial_j J_i^a(x_0-\Delta)\right)\left(\partial_i J_j^a(x_0+\Delta) - \partial_j J_i^a(x_0+\Delta)\right)\right].
\end{aligned}
\tag{22}
$$

Here, the first term is obtained from the term $\left(\partial_i A_4\right)^2 = \mu^2 \left(\partial_i \Phi\right)^2$. This term is the singular one. It can be verified that the term tends to infinity as $1/r^2$, when r goes to zero. To do this one needs to use parametrization of the function $\Phi^a(x) = (x-x_0)^a/r$.The quantity $1/r^2$ may be considered to be of the order of the ultraviolet cuttoff squared for a small r^2. Therefore the first term gives the contribution to the perturbative part of the vacuum energy density [1, 6]. Since we are interested in non-perturbative part of the vacuum energy density then we shall not discuss the first term.We recall that we assumed that the perturbative conributions into v.e.v. of $\theta_{\mu\mu}$ are subtracted.

The non-perturbative part of the vacuum energy is associated with the second term in (22) and can be written as

$$
\begin{aligned}
< 0|\theta_{\mu\mu}|0 >= \lim_{\Delta\to 0} b/64\pi^2 \left(\partial_i J_j^a(x_0-\Delta) - \partial_j J_i^a(x_0-\Delta)\right) \\
\left(\partial_i J_j^a(x_0+\Delta) - \partial_j J_i^a(x_0+\Delta)\right).
\end{aligned}
\tag{23}
$$

It is convinient to rewrite eq.(23) in another form, making use the eq.(2)

$$
F_{ij} = \partial_i J_j^a - \partial_j J_i^a + \varepsilon^{abc} J_i^b J_j^c = 0,
$$

and we arrive at

$$
\begin{gathered}
< 0|\theta_{\mu\mu}|0 >= \\
\lim_{\Delta\to 0} b/64\pi^2 \left\{\left(\partial_i \Phi^a(x_0-\Delta)\partial_i \Phi^a(x_+\Delta)\right)^2 \left(\Phi^b(x_0+\Delta)\Phi^b(x_0-\Delta)\right)^2 - \right. \\
- \left(\partial_i \Phi^a(x_0-\Delta)\partial_j \Phi^a(x_0-\Delta)\right) \\
\left.\left(\partial_i \Phi^b(x_0+\Delta)\partial_j \Phi^b(x_0+\Delta)\right)\left(\Phi^c(x_0-\Delta)\Phi^c(x_0+\Delta)\right)^2\right\}.
\end{gathered}
\tag{24}
$$

Here the terms of the type $\partial_i \Phi^a(x_0-\Delta)\Phi^a(x_0+\Delta)$ were omitted because they tend to zero when Δ goes to zero. The quantity $\partial_i \Phi^a(x_0-\Delta)\partial_j \Phi^a(x_0+\Delta)$ is approximately equal to $\delta_{ij}/3\partial_k \Phi^a(x_0-\Delta)\partial_k \Phi^a(x_0+\Delta)$ and due to that we get

$$
\varepsilon_{vac}^n = \lim_{\Delta\to 0} b/384\pi^2 \left(\partial_i \Phi^a(x_0-\Delta)\partial_i \Phi^a(x_0+\Delta)\right)^2 \left(\Phi^b(x_0-\delta)\Phi^b(x_0+\Delta)\right)
\tag{25}
$$

Now it should be shown that the quantity ε_{vac} is scale independent constant. We can do it in two steps. As the fields Φ^a have been defined on

the surface of the sphere S^2 then at the first step we can place the fields $\Phi^a(x_0-\Delta)$ and $\Phi^a(x_0+\Delta)$ in one and the same point on the surface keeping the radius of the sphere constant. At the second step the radius tends to zero.It is convenient to introduce new variables Δ_i and $r^2 = \Delta_i^2$ holding x_0 fixed. Making use of the parametrization of the function $\Phi^a = \gamma\Delta^a/r$ we have

$$\left(\partial_i\Phi^a\right)^2 = -\frac{2}{r^2}\gamma^2. \tag{26}$$

Here γ is the normalization constant which is given below.

To calculate the quantity $\Phi^a(x_0-\Delta)\Phi^a(x_0+\Delta)$ one can use eq.(14) and obtain

$$\Phi^a(x_0-\Delta)\Phi^a(x_0+\Delta) = \Phi^b(x_0)\Phi^c(x_0)\left(P\ \exp\left(-\int\limits_{x_0-\Delta}^{x_0+\Delta} dz_i\ J_i\right)\right)_{bc}. \tag{27}$$

We can write eq.(27) in a more suitable form. To this end one expands of exponential function in eq.(27) in powers of small Δ to second order and gets that the first oder term is equal to zero due to antisymmetric tensor ε_{abc} and the second order term contains $\Phi^b\Phi^c$ times $\Phi^b\Phi^c$ and thus one gets $\left(\Phi^2\right)^2$ =1. Therefore eq.(27) can be written as follows

$$\Phi^a(x_0-\Delta)\Phi^a(x_0+\Delta) = \left(P\ \exp\left(-\int\limits_{x_0-\Delta}^{x_0+\Delta} dz_i\ J_i\right)\right)_{aa}. \tag{28}$$

The contour of integration should be closed around the sphere with the fixed radius. Then substituting (26) and (28) into (25) we have

$$\varepsilon^n_{vac} = \lim_{r\to 0} b/96\pi^2\left(\frac{\gamma^2}{r^2}P\ \exp\left(-\oint dz_i\ J_i\right)\right)^2 \tag{29}$$

To calculate the integral around the circle use is made of the identy $\left(J_i^a t^a\right)^{bc} = \Phi^b\partial_i\Phi^c - \partial_i\Phi^b\Phi^c$ which yeilds

$$\oint dz_i\ J_i^{bc} = \oint d\Phi^c\,\Phi^b - d\Phi^b\,\Phi^c = 2S^{bc} \tag{30}$$

The area S^{bc} is enclosed by the circle.The third axis may be oriented normally to the area S^{bc} in the colour space.The field Φ^3 is fixed in such a way that fields Φ^1 and Φ^2 are normalized as $\left(\Phi^i\right)^2 = \lambda, i = 1, 2$. In this case the area S^{12} equals $\pi\lambda$ and the integral around the circle is $2\pi\lambda$. For ε^n_{vac} we have

$$\varepsilon^n_{vac} = b/96\pi^2\left(\frac{\gamma^2}{r^2}\exp(-2\pi\lambda)\right)^2. \tag{31}$$

Here ε_{vac} is obtained on the scaling mass $\mu^2 = 1/r^2$. Now we should get the vacuum energy density corresponding to $r^2 = 1/M^2$. Since the fields were defined on the scaling mass μ^2 they can be rewritten in terms of the bare fields. Then we have $J_i = Z_3^{1/2} J_i$ and $\Phi_b = Z^{1/4}\Phi_b$. In this case the relation between λ and unrenormalized constant λ_b is written as $\lambda = Z^{-1/2}\lambda_b$. Besides we have to define a relation $< 0|\theta_{\mu\mu}|0 >= Z_3^{-1} < 0|\theta_{\mu\mu}|0 >_b$. Clearly the constant γ^2 in eq.(26) is $Z_3^{-1/2}$, and we get

$$\varepsilon^n_{vac} = b/96\pi^2 \left(\mu^2 \exp\left(-2\pi\lambda_b Z_3^{-1/2}(\mu^2)\right)\right). \tag{32}$$

The value ε_{vac} is scale independent only if

$$\mu^2 \frac{d\varepsilon^n_{vac}}{d\mu^2} = 0.$$

Using (32) and (18) we obtain that if

$$\lambda_b = 8\pi/(g_b^2 b),$$

then ε^n_{vac} is scale independent. Now we obtain the final result

$$\varepsilon^n_{vac} = b/96\pi^2 \left(M^2 \exp\left(-16\pi^2/bg_b^2\right)\right)^2. \tag{33}$$

The quantity ε^n_{vac} is scale independent and rises as N which is agremeant with the familiar result [1].

Notice that λ_b is a large quantity ($\lambda \gg 1$) due to $g_b^2(M^2)/4\pi \ll 1$,therefore $\Phi_3^2 = 1 - \lambda_b$ may be negative quantity and this, in its turn, implies that the condition $(\Phi^a)^2 = 1$ is not the equation of a sphere in view of quantum effects, i.e. the monopoles are absent in the scale independent quantum theory.

5 Conclusions.

In the present paper we have calculated the non-perturbative part of the vacuum energy density in pure Y-M theory which is determined by the static vacuum fluctuations in one-loop level. The result was obtained by using the physical ideas which are derived in the special case of sigma models [6, 7].

It was shown that the vacuum energy density in Y-M theory is the scale independent quantity, which corresponds to the topological charge equal to zero.

The method of regularization is proposed which is based on the critical assumption that at each point of the space in which a product of two operators can be replaced by a sphere S^2 having a small radius which is set to zero at the very end of the calculation and the quantum fields are separated in diffirent point on the surface of the sphere. The calculation have been carried out using functional integral but can be done by the operator method. For this aim the vacuum state has been constructed.

Acknowledements.

The author wishes to thank V.A. Novikov, A.D. Mironov, Yu.M. Makeenko for useful discussions. The paper was partially supported by the Russian Foundation of Fundamental Reseach (grant No 98-02-17316).

References

[1] V.A.Novikov, M.A.Shifman, A.I. Vainshtein and V.I.Zakharov, Phys.Rep., 116 (1989) 103.

[2] W.A. Bardeen, B.W. Lee and R.E. Shrock, Phys.Rev., D14 (1976) 985.

[3] B.Rosenstein, B.J. Warr, S.H. Park, Phys.Rep., 205 (1991) 59.

[4] G.'t Hooft, Phys.Rev., D14 (1976) 3432.

[5] C.G.Callan, R.Dashen, D.J.Gross, Phys.Rev., D17 (1978) 2717.

[6] V.G.Ksenzov, Phys.Lett., B367 (1996) 237.

[7] V.G.Ksenzov, Phys.Lett., B410 (1997) 52.

[8] R.Rajaraman,in Solitons and Instantons. North-Holland Publishing Company (1982) 414.

[9] M.K.Prasad, Physica 1D (1980) 167.

[10] V.N.Gribov, Nucl.Phys., B139 (1978) 1.

[11] A.A.Migdal, M.A.Shifman, Phys.Lett., B114 (1982) 445; M.B.Voloshin, K.A.Ter-Martirosyan, in Theory of Gauge Interaction of Elementary Particles, Energoatomizdat, Moscow (1984) 277.

[12] C.Itzykson, J.B.Zuber, in Quantum Field Theory, vol.2, McGraw-Hill (1980).